Yiddish of the Day

"Di velt is oykh nisht farshafn gevorn in eyn tog" =

די וועלט איז אויך
נישט פֿאַרשאַפֿן
געוואָרן אין איין טאָג

"the world was also not created in one day" =

- Do the SETS
  - → At 9/22 (41%)
  - Need 85% for EC
  - → Due Tuesday

Recall: Given 2 linear maps

$g_1 : V \to W$

$g_2 : V \to W$

get $\to$ linear $\underline{g_1 \otimes g_2} : V \otimes V \to W \otimes W$

such that $\underline{g_1 \otimes g_2}(v_1 \otimes v_2) = \underline{g_1}(v_1) \otimes \underline{g_2}(v_2)$

Prop: Let $g : V \to W$ linear. Then there exists a ! linear map

$$\underline{\Lambda^2(g)} : \underline{\Lambda^2(V)} \to \underline{\Lambda^2(W)}$$

such that $\underline{\Lambda^2(g)}(v_1 \wedge v_2) = \underline{g}(v_1) \wedge \underline{g}(v_2)$

(Rmk: True for more general K.)

Pf) Define the function $V \times V \to \Lambda^2(W)$

$(v_1, v_2) \to g(v_1) \wedge g(v_2)$

Note that $(v,v) \to g(v) \wedge g(v) = 0$

so this is alternating

Now $(v_1 + v_1', v_2) \to g(v_1 + v_1') \wedge g(v_2)$

$= (g(v_1) + g(v_1')) \wedge g(v_2)$

$= g(v_1) \wedge g(v_2) + g(v_1') \wedge g(v_2)$

$(cv_1, v_2) \mapsto g(cv_1) \wedge g(v_2)$

$= c\, g(v_1) \wedge g(v_2)$

$= g(v_1) \wedge c\, g(v_2)$

$= g(v_1) \wedge g(cv_2)$

$\to \exists!$ linear map $\Lambda^2(V) \to \Lambda^2(W)$ sending $v_1 \wedge v_2 \to g(v_1) \wedge g(v_2)$ $\square$

Just like before, given

$$V \xrightarrow{g_1} W \xrightarrow{g_2} Z$$

we have $\underline{\Lambda^k(g_2 \circ g_1)} = \underline{\Lambda^k(g_2) \circ \Lambda^k(g_1)} : \underline{\Lambda^k(V)} \longrightarrow \underline{\Lambda^k(Z)}$

and for the identity map $V \xrightarrow{id} V$ we have

$$\underline{\Lambda^k(id)} = id_{\Lambda^k(V)} : \underline{\Lambda^k(V)} \longrightarrow \underline{\Lambda^k(V)}$$

ex) $V = \mathbb{C}[t]_{\leq 2}$ with basis $\mathcal{B} = (1, t, t^2)$

$g : V \longrightarrow V$ be

$g(f(t)) = \underline{f'(t) + 3f(t)}$

i) Find $[g]_{\mathcal{B}} = \boxed{\begin{pmatrix} 3 & 1 & 0 \\ 0 & 3 & 2 \\ 0 & 0 & 3 \end{pmatrix}}$

ii) Compute the matrix of $\Lambda^2 g : \Lambda^2(V) \longrightarrow \Lambda^2(V)$ with respect to basis

$\mathcal{C} = (\underline{1 \wedge t, 1 \wedge t^2, t \wedge t^2})$

- $\Lambda^2(g)(1 \wedge t) = g(1) \wedge g(t) = 3 \wedge (1 + 3t)$

$$= 3\wedge 1 + 3\wedge 3t$$

$$= \cancel{3(1\wedge 1)} + \underbrace{9(1\wedge t)}$$

- $$\Lambda^2(\varphi)(1\wedge t^2) = \varphi(1)\wedge\varphi(t^2) = 3\wedge(2t+3t^2)$$
$$= 3\wedge 2t + 3\wedge 3t^2$$
$$= \underline{6(1\wedge t) + 9(1\wedge t^2)}$$

- $$\Lambda^2(\varphi)(t\wedge t^2) = \varphi(t)\wedge\varphi(t^2) = (1+3t)\wedge(2t+3t^2)$$
$$= 1\wedge 2t + 1\wedge 3t^2 + 3t\wedge 3t^2$$
$$= 2(1\wedge t) + 3(1\wedge t^2) + 9(t\wedge t^2)$$

$$[\Lambda^2\varphi^{c}] = \begin{pmatrix} 9 & 6 & 2 \\ 0 & 9 & 3 \\ 0 & 0 & 9 \end{pmatrix}$$

ex2) What about the map $\Lambda^3(\varphi): \Lambda^3(V) \to \Lambda^3(V)$.

Note $\dim V = 3$ so $\dim(\Lambda^3(V)) = \underline{\ 1\ }$

So this map $\Lambda^3(\varphi)$ will just be scaling by a #.

<u>What #?</u> Have basis

$e = (1 \wedge t \wedge t^2)$

<u>Compute</u>: $\Lambda^3(\varphi)(1 \wedge t \wedge t^2) = \varphi(1) \wedge \varphi(t) \wedge \varphi(t^2)$

$= 3 \wedge (1+3t) \wedge (2t+3t^2)$

$= 3 \wedge 3t \wedge (2t+3t^2)$

$= 3 \wedge 3t \wedge 2t + 3 \wedge 3t \wedge 3t^2$

$$= \cancelto{0}{18(1 \wedge t \wedge t)} + 3^3(1 \wedge t \wedge t^2)$$

$$= 3^3(1 \wedge t \wedge t^2)$$

$$= \underline{27}(1 \wedge t \wedge t^2)$$

$$= \det(g)(1 \wedge t \wedge t^2)$$

Def: Let V be n-dim vs, and let

$\varphi: V \to V$ be a linear map.

We define the **determinant** of φ to be

the scalar multiple on which $\Lambda^n(\varphi)$ acts.

(that is determinant is the ! number st

$$\Lambda^n(\varphi)(v_1 \wedge \dots \wedge v_n) = \det(\varphi)\, v_1 \wedge \dots \wedge v_n \text{)}$$

Thm: Let $\varphi: V \to V$ and $\gamma: V \to V$ be 2 linear maps.

Then $\det(\varphi\gamma) = \det \varphi \cdot \det(\gamma)$

Pf) $\Lambda^n(\varphi\gamma) = \Lambda^n(\varphi)\circ\Lambda^n(\gamma) = \det(\varphi)\det(\gamma)\, v_1\wedge\cdots\wedge v_n$

$$\Lambda^n(\varphi\gamma)(v_1\wedge\cdots\wedge v_n)$$

$$\shortparallel$$

$$\det(\varphi\gamma)\, v_1\wedge\cdots\wedge v_n$$

Cor: i) We can compute the determinant of φ using any basis.

HW ii) If φ is an isomorphism then $\det(\varphi^{-1}) = (\det(\varphi))^{-1}$

Pf) Recall that a change of basis is just a multiplication

$$P\varphi P^{-1} \rightsquigarrow \Lambda^n(P\varphi P^{-1}) = \Lambda^n(P)\circ\Lambda^n(\varphi)\circ\Lambda^n(P^{-1})$$

$$\rightsquigarrow \det(P\varphi P^{-1}) = \det(P)\det(\varphi)\det(P)^{-1}$$

$= \det(B)$

ex) $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ viewed as a linear map $\mathbb{R}^2 \to \mathbb{R}^2$

Then $\Lambda^2(A)(e_1 \wedge e_2) := A(e_1) \wedge A(e_2)$

$$= (ae_1 + ce_2) \wedge (be_1 + de_2)$$

$$= \cancel{ae_1 \wedge be_1} + ae_1 \wedge de_2 + ce_2 \wedge be_1 + \cancel{ce_2 \wedge de_2}$$

$$= ad\, e_1 \wedge e_2 + bc\, e_2 \wedge e_1$$

$$= (ad - bc)\, e_1 \wedge e_2$$

$$= \det(A)\, e_1 \wedge e_2$$

Back to Reality

Some (hopefully) famillar definitions

Def: Let $T: V \to V$ be linear map.

We say of $v \in V$ is an eigenvector if

$T(v) = \lambda v$ (call this $\lambda \in F$ an eigenvalue)

$\Big[\, Tv - \lambda v = 0$
$\quad \overset{"}{} (T - \lambda\, id)(v) \,\Big]$

Q: How to find eigenvectors

→ If $v \in V$ is an eigenvector then

$(T - \lambda \mathrm{id}) v = 0$

$\iff (T - \lambda \mathbb{1})$ has nontrivial kernel

$\iff (T - \lambda \mathbb{1})$ is not isomorphism

(HW)

$\iff \det (T - \lambda I) = 0$

$\Rightarrow$ **Prop / Def:** Let $T: V \to V$ be linear. Define

$$c_T(x) := \det(T - x\mathbb{1})$$

Then the eigenvalues of T are the roots of $c_T(x)$

Def: Let $\lambda \in \mathbb{F}$ be an eigenvalue of T. Write

$$E_\lambda := \ker(T - \lambda I)$$ = the eigenvectors corresponding to λ

"eigenspace"

• We call $\dim(E_\lambda) :=$ geometric multiplicity

$(x-2)^3(x+1)^3$

Rmk: There is another notion of multiplicity

The "algebraic multiplicity"

If λ is root of $c_T(x)$ then

$$c_T(x) = (x-\lambda)^d p(x)$$

This d is the algebraic multiplicity

Fact: geometric mult $\leq$ algebraic mult

There is a whole story here about

- minimal polynomials
- characteristic polynomials

} and their interplay

- Diagonalizability (some more on this Wed)
- Jordan Form (with "generalized eigenvectors")

- "Cayley - Hamilton" thm

- All of these are really important, and should be looked into if one is serious about learning linear algebra.

- Next time $\longrightarrow$ inner product spaces / adjoint (M)
  Spectral Thm (W)